Strength of Materials

An anglicized version for the students of the Higher Institute of Technological Studies of Rades (ISET Rades)

Translated to English & illustrated by mechanical engineering student:

Amen Nafti

Project supervised by university Professor:

Mr. Adel Hedhli

Introduction

Strength of materials, also called “mechanics of materials” is a necessary knowledge for mechanical engineering students. Since it is known that English is an international language for communication, we offered this simplified course to be given as an addition to students who are willing to take part in international programs, study abroad or work abroad.

This work is meant to be available as an open source course, as digital and paperback versions. These can help explain some phenomena, translate technical terms and get used to other notations.

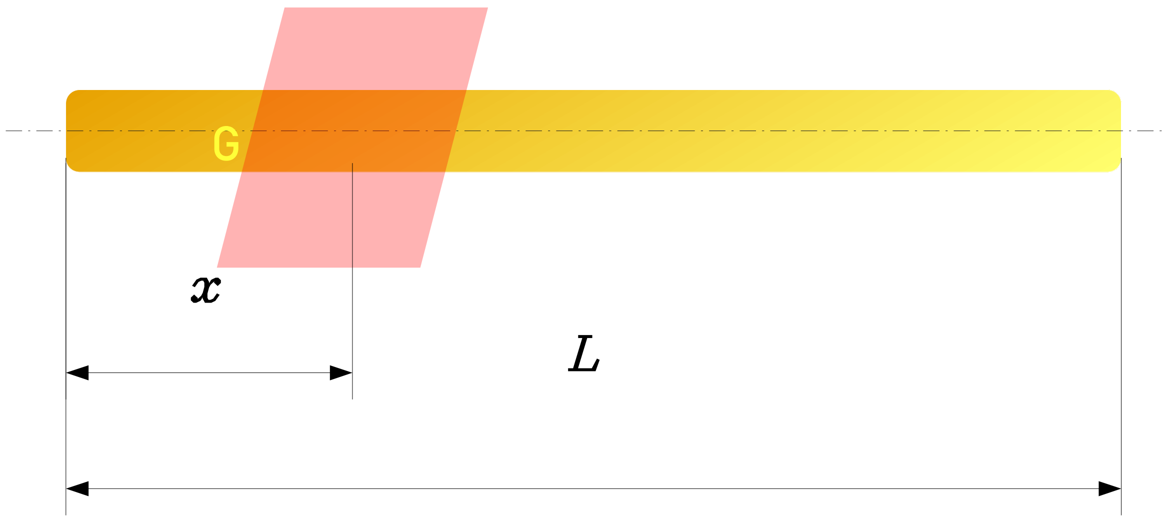
Strength of materials is widely used in mechanical and civil fields. Understanding this subject allows technicians and engineers to better analyze systems and to build more reliable products. Mechanisms are succumb to forces, torques, and loads, exposing them to deformation. The latter (deformation) should be thoroughly studied, since as we have already mentioned how the objects and beams that we’re studying take part in a mechanism, thus, every deformation counts. The slightest mistake in calculation or choice of either material or dimension can lead the system to fail. Reliable systems are demanded and essential for a sustainable future for the industry.

Cohesion Torsors

Torsors are an algebraic entity that simplifies the denotation of various elements. In our study, **cohesion torsors** represent the internal solicitations in beams due to external forces and torques. It can reduce the writing of six components of forces and torques, three each, in one entity as follows:

$$\{\tau_{coh\,II/I}\}_G = \begin{Bmatrix} Fx & Mt \\ Fy & Mfy \\ Fz & Mfz \end{Bmatrix}_G$$

We begin with splitting our beam into parts. G is the centre of inertia of the section we chose to work on. Our first section has the length of x



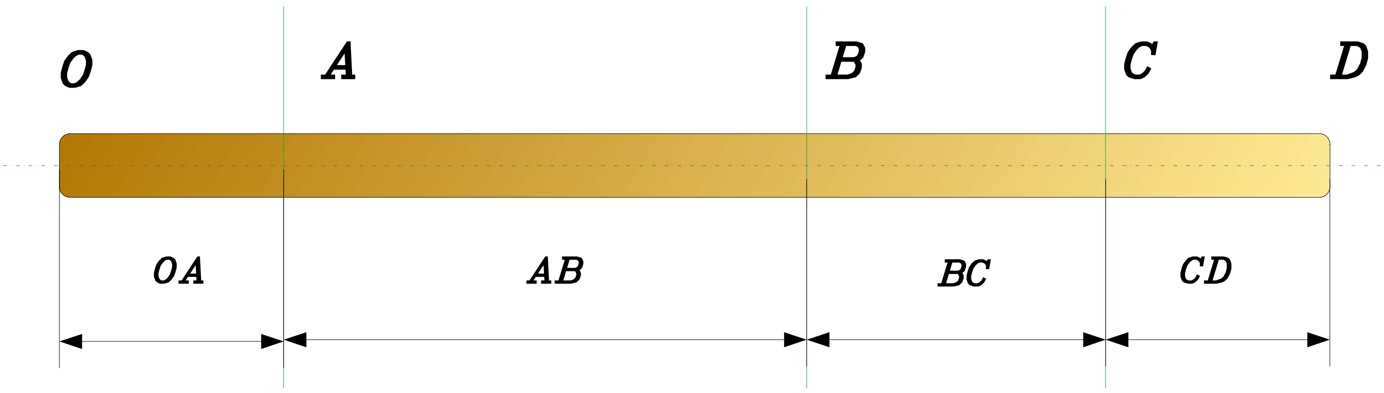
After splitting the beam, we choose a direction to work on. If we are working on the left, the resulting cohesion torsor is minus the sum of the torsors on the left of the section in the point G.

$$\{\tau_{coh\,II/I}\}_G = -\begin{Bmatrix} Fx & Mt \\ Fy & Mfy \\ Fz & Mfz \end{Bmatrix}_G$$

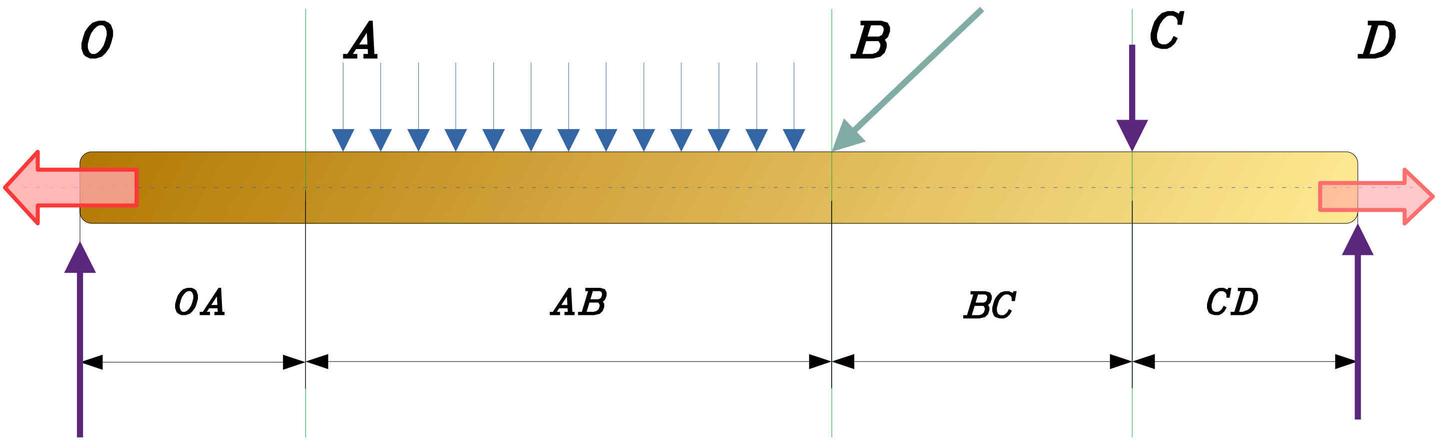
If we are working on the right, the resulting cohesion torsor is the sum of the torsors on the right of the section in the point G.

$$\{\tau_{coh\,II/I}\}_G = +\begin{Bmatrix} Fx & Mt \\ Fy & Mfy \\ Fz & Mfz \end{Bmatrix}_G$$

Cohesion torsors express the efforts and torques applied to the beam. Each component will help draw the diagrams of normal and shear forces, torque and bending moments. It simplifies the way we describe whatever happens to our system in any zone we got.



We will apply some forces and moments to the beam. We will have an outcome as follows:



In each zone, we transfer the static torsors of each point to the point G. While resulting forces do not change, moments are transferred and can change. We will work on the leftmost zone (OA):

$$\{\tau_{\vec{O}}\}_O = \begin{Bmatrix} 0 & -Mt \\ F_O y & 0 \\ 0 & 0 \end{Bmatrix}_O$$

We will transfer the torsor from the point O to the point G, the centre of inertia of the section:

$$\{\tau_{\vec{O}}\}_G = \begin{Bmatrix} 0 & -Mt \\ F_O y & 0 \\ 0 & 0 \end{Bmatrix}_G = \begin{Bmatrix} \overrightarrow{F_O} \\ M_G(\vec{O}) \end{Bmatrix}_G = \begin{Bmatrix} 0 & -Mt \\ F_O y & 0 \\ 0 & -Fy.x \end{Bmatrix}_G$$

where: $M_G(\vec{O}) = M_O(\vec{O}) + \overrightarrow{GO} \wedge \overrightarrow{F_O}$

Thus:

$$\{\tau_{coh\,II/I}\}_G = -\{\tau_{\overrightarrow{F_O}}\}_G = -\begin{Bmatrix} 0 & -Mt \\ F_O y & 0 \\ 0 & -Fy.x \end{Bmatrix}_G = \begin{Bmatrix} 0 & Mt \\ -F_O y & 0 \\ 0 & F_O y.x \end{Bmatrix}_G$$

Tension & Compression

When two opposing forces are applied to the beam, directed **outwards,** tend to **stretch** it the beam is succumb to **tension**

A beam undergoing tensile efforts has a cohesion torsor in its centre of inertia as follows:

$$\{\tau_{cohII/I}\}_G = \begin{Bmatrix} N & 0 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}_G$$

Normal Stress

Solicitations like tension invoke **normal stress** denoted by **σ** (a pressure whose direction is perpendicular to the cross-section of the beam) where:

$$\sigma = \frac{F}{S}$$

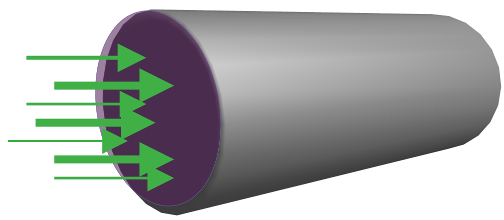


Fig 1

σ: normal stress in N/mm^2 or MPa

F: normal force applied to the cross-section in N

S: cross-section area in mm^2

Tensile Test

Tensile test consists of an application of **tensile load** to a specimen that will keep getting stretched until reaching a maximum elongation. That's when the specimen eventually fails and breaks in two parts. This test aims at determining some mechanical characteristics of the studied material, such as:

Percentage elongation denoted **δ_n** where: $\delta_n = \frac{l_u - l_0}{l_0} \times 100$

with $l_u = l_0 + \Delta l$

Percentage reduction in area denoted **Z** where: $Z = \frac{S_0 - S_u}{S_0} \times 100$

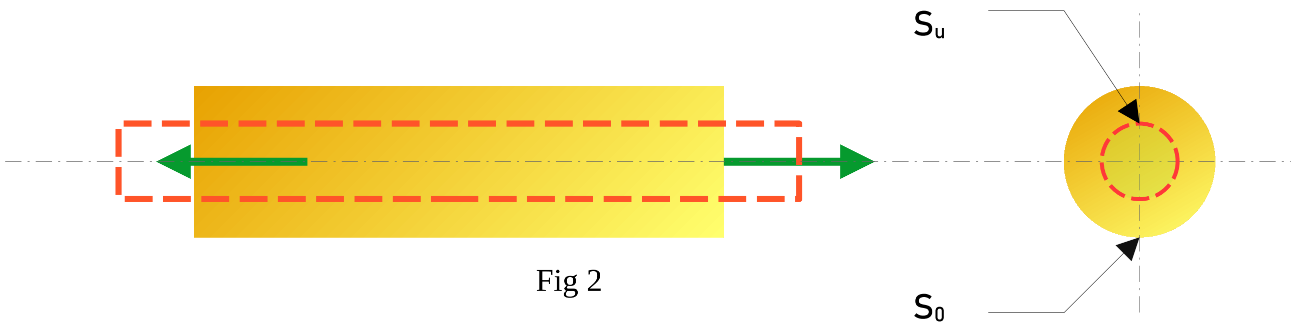


Fig 2

Young's modulus or **the modulus of longitudinal elasticity**, denoted **E**, can be graphically determined from the **conventional stress-strain curve**; it is the slope of the linear zone (OA) of the curve: $E = \frac{\sigma}{\epsilon}$

Tensile strength denoted **σ_m** where: $\sigma_m = \frac{F_{max}}{S_0}$

Rupture strength denoted **σ_r** where: $\sigma_r = \frac{F_r}{S_0}$

Elasticity limit denoted **σ_e** where: $\sigma_e = \frac{F_e}{S_0}$

Elongation denoted **ε** where: $\epsilon = \frac{\Delta l}{l_0}$

After reaching the maximum load or stress, **necking** occurs. Necking is the heavy diminution of the cross-section area of the specimen due to uniformity in the distribution of strength along its length.

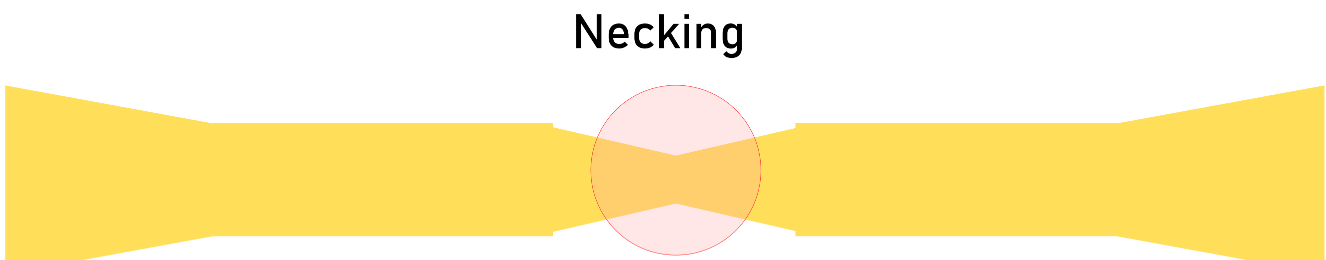


Fig 3

A device is used to continuously record the evolution of load and elongation. A **force-displacement diagram** is drawn as follows:

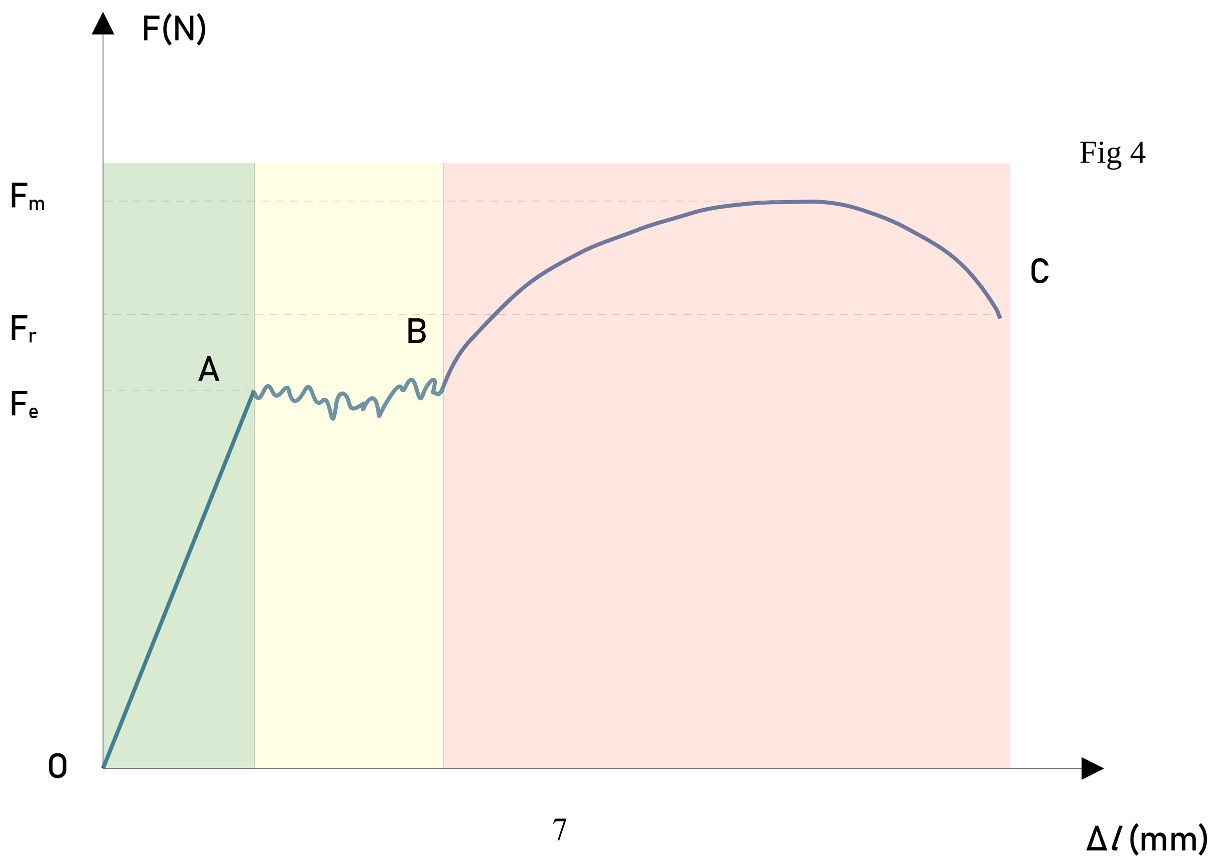


Fig 4

We can then draw the **stress-strain** diagram:

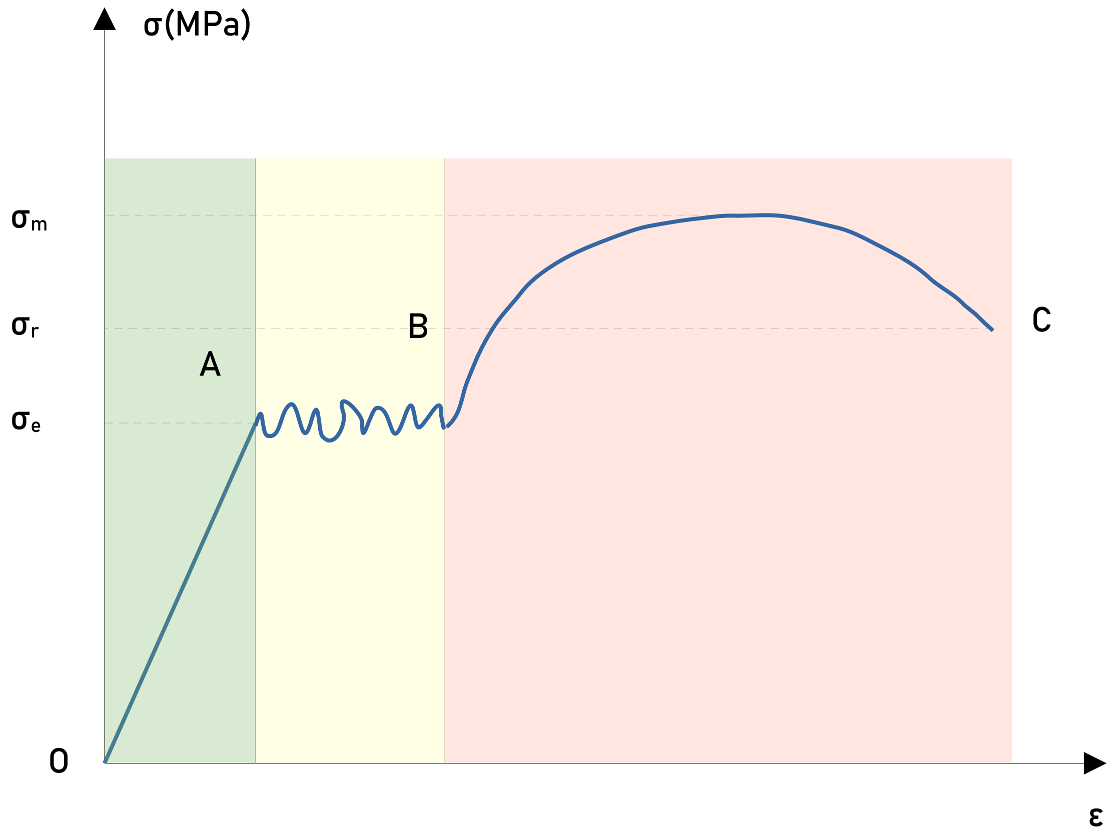


Fig 5

Zone OA represents a linear variation of the stress. Here, stress and strain are **proportional** and give us the relation: $E=\frac{\sigma}{\epsilon}$ where E is **Young's modulus** or the modulus of longitudinal elasticity. The zone is limited by the elasticity limit σ_e.

Zone AB represents the **yield plateau**. It is when the tested specimen endures an increase in strain with a slight or no increase at all of stress. The material starts behaving **plastically**.

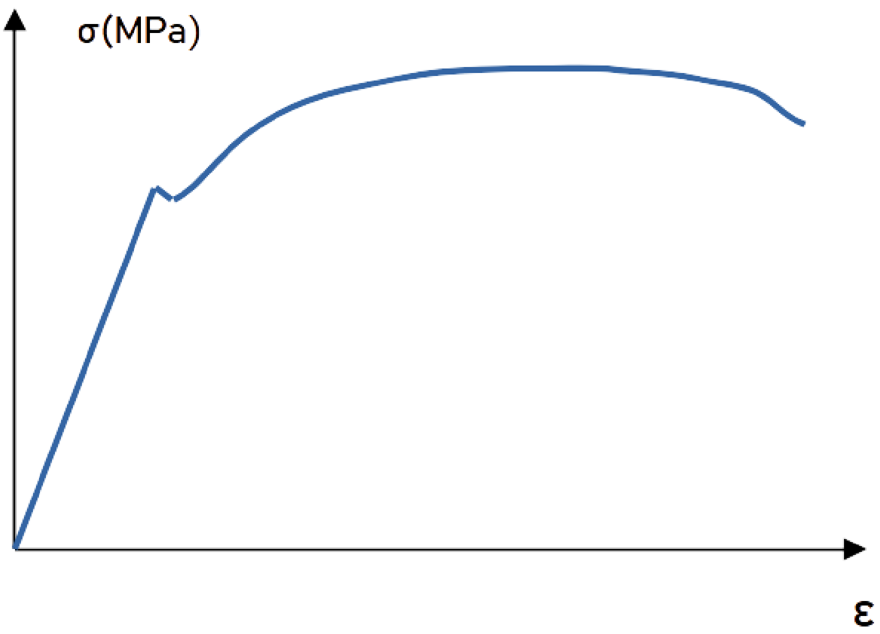
Zone BC corresponds to the hardening zone where both strain and stress grow until reaching a maximum value of stress σ_m then a lower value of rupture strength σ_r and a maximum elongation ε.

Hooke's Law

Since internal stress invokes deformation, it is possible to track the latter and extract some important material characteristics through curves.

For the normal stress σ, we can trace the **stress-strain** graph: σ=f(ε):

Fig 6



From the graphs, we can extract **Young's modulus** denoted **E** and is calculated as follows: $E=\frac{\sigma}{\epsilon}$

Resistance Condition

In order to resist failure, we have set a limit value of stress that shouldn't be surpassed. $\sigma_{max} \le R_{pe}$ where: $Rp_e=\frac{R_e}{s}=\frac{\sigma_e}{s}$ s being a security coefficient.

Rigidity Condition

Rigidity condition sets a limit to the longitudinal deformation of the beam, where: $\Delta l \leq \Delta l_{\lim}$.

Stress Concentration

Beams in real life are no perfect, uniform structures. They definitely do have variations in dimensions and geometry. Grooves, threads and holes are considered inconsistencies, which make stress concentrated on the weakest section of the studied shaft or beam. This impacts the resistance condition by multiplying the tensile strength by a constant value **k_t.**

$$k_t \sigma_{max} \leq R_{pe}$$

The latter is chosen from an abacus as follows:

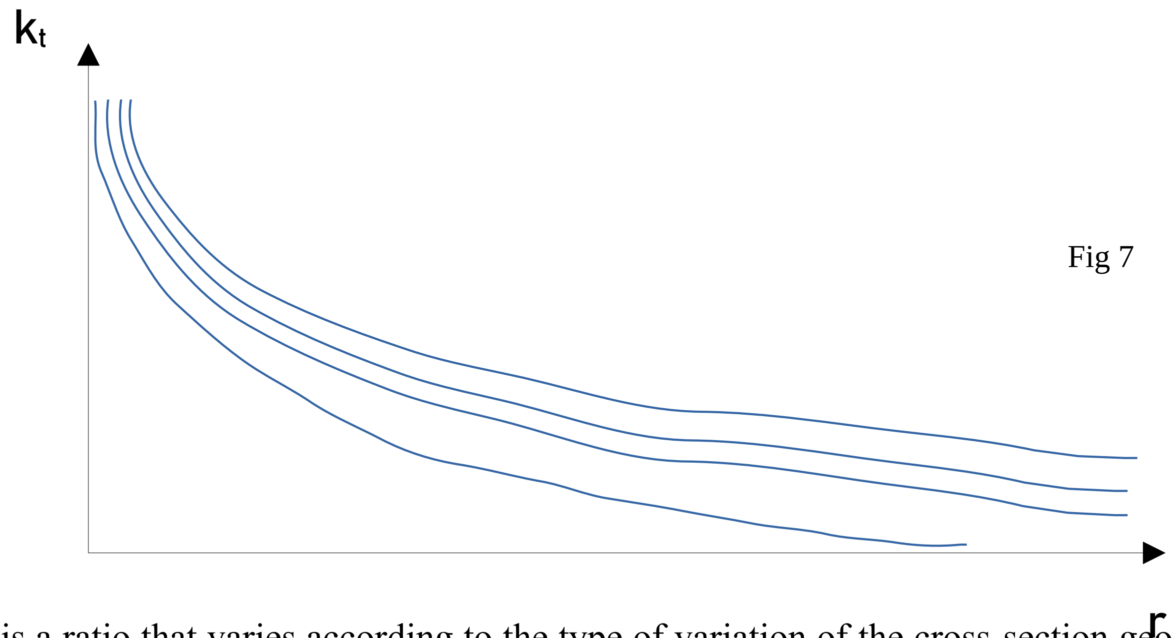


Fig 7

r is a ratio that varies according to the type of variation of the cross-section geometry (i.e. groove, thread, etc.)

Compression

A beam is succumb to **compression** when two opposing forces, directed to**ward** it tend to **compress** it.

A beam undergoing compressive efforts has a cohesion torsor in its centre of inertia as follows:

$$\{\tau_{cohII/I}\}_G = \begin{Bmatrix} -N & 0 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}_G$$

Resistance Condition

Similar to tension, in order to resist failure, we have set a limit value of stress that shouldn't be surpassed. $\sigma_{max} \leq Rp_e$

Rigidity Condition

Rigidity condition sets a limit to the longitudinal deformation of the beam, where: $|\Delta l| \leq \Delta l_{\lim}$. We take the absolute value since the variation of length is negative in compression.

For longer beams, compression leads to buckling. This phenomenon has its own calculations depending on the length of the beam. (View Buckling chapter)

In compression, the variation of length is negative because the compressive forces squeeze the beam longitudinally, however, the cross-section area increases instead. In calculations, we consider that $S_0 = S_u$

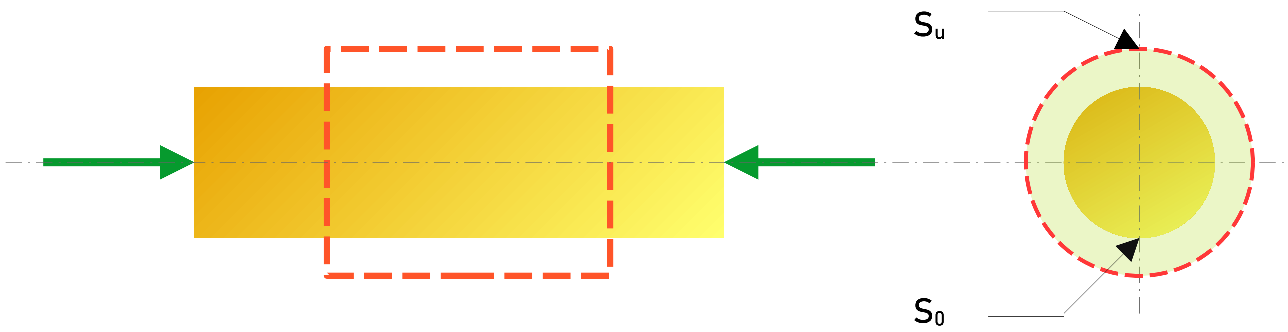


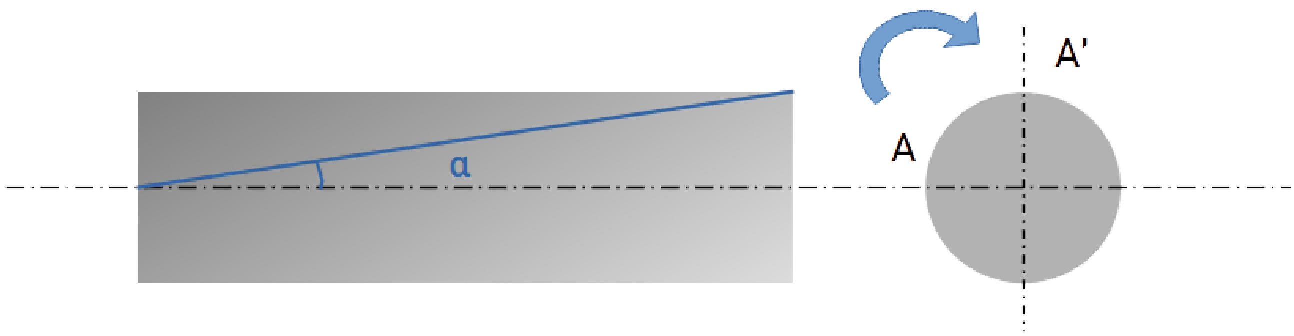
Fig 8

Torsion

A beam is said to undergo **torsion** when a twisting moment acting about its axis named **torque** tends to twist it. The cohesion torsor of this solicitation is written as follows:

$$\{\tau_{cohII/I}\}_G = \begin{Bmatrix} 0 & M_T \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}_G$$

Torsion is characterized by an **angle of twist** denoted **α** (also **ϕ**) that describes the angle between two sections along the beam.



α (rad) is calculated as follows: $\alpha = \dfrac{M_T L}{G J}$

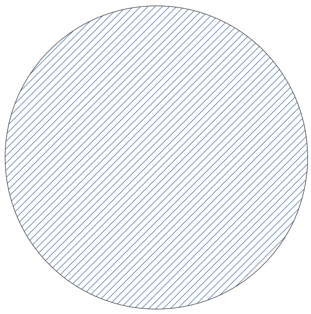
T: torque (N.mm)

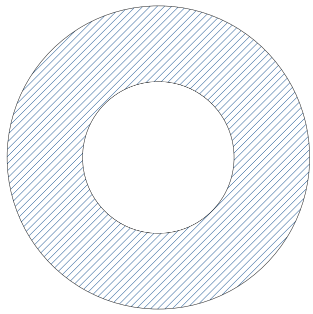
L: length of the beam (mm)

G: shear modulus (Coulomb's modulus) (MPa)

J (or I_G): polar moment of inertia (mm^4)

The polar moment of inertia is calculated according to the shape of the cross-section of the beam. We usually work on circular sections.





$J=\frac{\pi D^4}{32}$	$J=\frac{\pi\left(D^4-d^4\right)}{32}$

Another angle is also calculated, and it is **the unitary angle of torsion** denoted **θ**, used to determine the condition of rigidity and is calculated as follows:

Resistance Condition

Stress in torsion is tangent to the cross-section. In order to resist efforts, the internal shear stress τ should stay below a set value. The latter is a practical sliding limit **Rp_g** measured in MPa and is: $Rp_g=\frac{R_g}{s}$

s is a security ratio and **R_g** is a sliding limit.

The condition of resistance becomes:

$$\tau_{max}\leq Rp_g \Leftrightarrow \frac{M_T R}{J}\leq Rp_g$$

Dimensioning the beam from the condition of resistance:

$$D \geq \sqrt[3]{\frac{16\,T}{\pi\,Rp_g}}$$

In case of concentrated stress:

$$D \geq \sqrt[3]{\frac{16\,K_t\,M_T}{\pi\,Rp_g}}$$

Deformation

As an equivalent to the condition of rigidity in tension and compression, deformation has its rule in torsion. It uses the unitary angle of twisting θ and compares it to a limit value: $\theta \leq \theta_{\lim.}$

Dimensioning the beam from this condition:

$$D \geq \sqrt[4]{\frac{32\,T}{\pi\,G\,\theta_{\lim.}}}$$

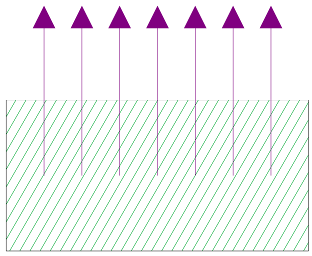
After calculating diameters from both conditions, choose the greater value among them.

Shearing

A beam is said to undergo shearing when a **shearing stress** tends to split it. The cohesion torsor of this solicitation is written as follows:

$$\{\tau_{cohII/I}\}_G = \begin{Bmatrix} 0 & 0 \\ Fy & 0 \\ 0 & 0 \end{Bmatrix}_G$$

or $$\{\tau_{cohII/I}\}_G = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ Fz & 0 \end{Bmatrix}_G$$



It is experimentally impossible to realize a shearing test. We're meant to study the tangential efforts in a cross-section of the beam, since shearing is not affected by its length. Shearing forces practically imply bending moments as the cohesion torsors showed. Shearing stress leads to a different solicitation: flexure (or bending).

Shear Stress

Shearing is associated with a **shear stress** denoted by **τ** (a pressure whose direction is tangent or parallel to the cross-section of the beam) where:

$$\tau = \frac{F}{S}$$

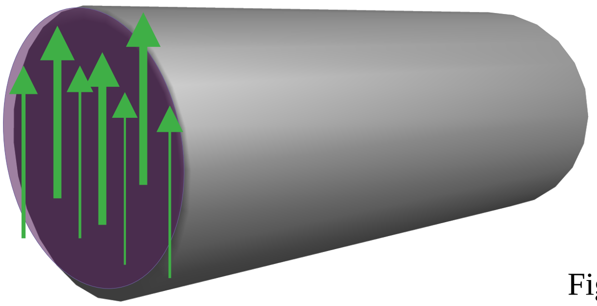


Fig 1

τ: shear stress in N/mm^2 or MPa

F: shear (tangential) force applied to the cross-section in N

S: cross-section area in mm^2

Shear Modulus

For the shear stress τ, we can trace the **stress-strain** graph: $\tau = f(\gamma)$:

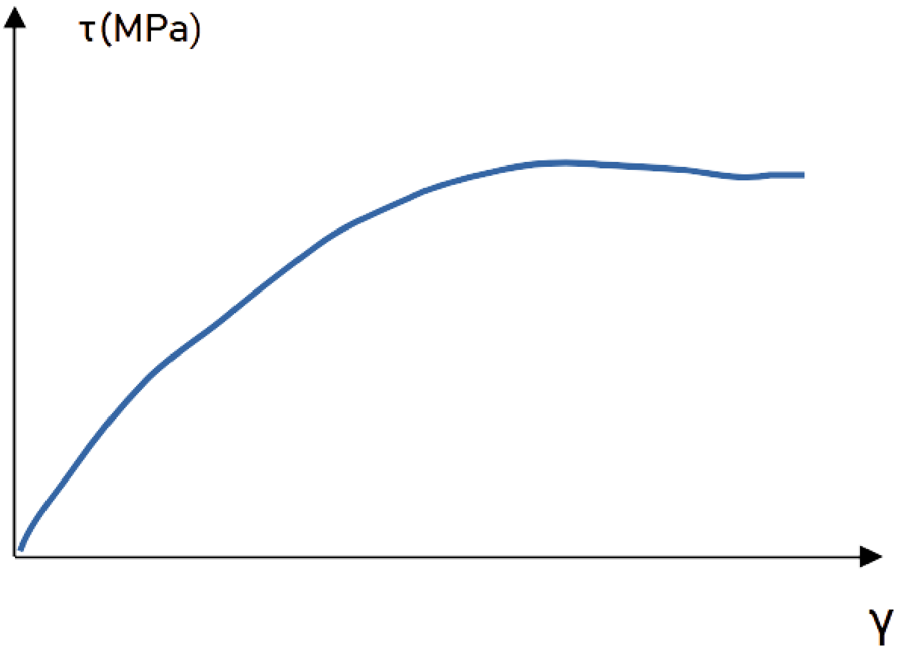


Fig 2

Coulomb's modulus or shear modulus represents **the modulus of transverse elasticity**.

From the graph, we can extract **Coulomb's modulus** denoted **G** and calculated as follows: $G=\frac{\tau}{\gamma}$

Resistance Condition

Every element of the surface bears a shearing effort included within the plane of the cross-section. The shear strain is thus tangent to the surface and is denoted τ where

$$\tau=\frac{F}{S}$$

Condition of resistance: $\tau \leq Rp_g$

Bending

We'll study the example of simple bending in this chapter.

A beam undergoes **bending** when exposed to a shearing effort and a bending moment as the cohesion torsor indicates:

$$\{\tau_{cohII/I}\}_G = \begin{Bmatrix} 0 & 0 \\ Fy & 0 \\ 0 & -Fy.\Delta x \end{Bmatrix}_G$$

or $$\{\tau_{cohII/I}\}_G = \begin{Bmatrix} 0 & 0 \\ 0 & Fz.\Delta x \\ Fz & 0 \end{Bmatrix}_G$$

Bending Test

Let's study whatever happens to a beam undergoing simple bending. The fibres of the tested beam will deform.

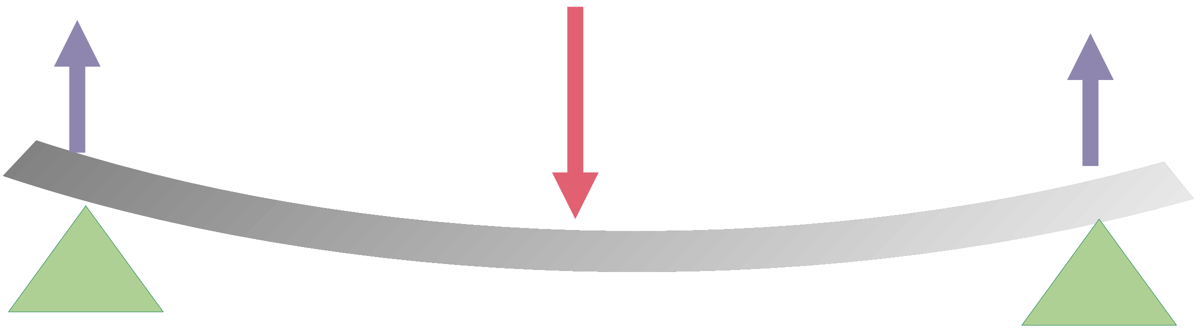
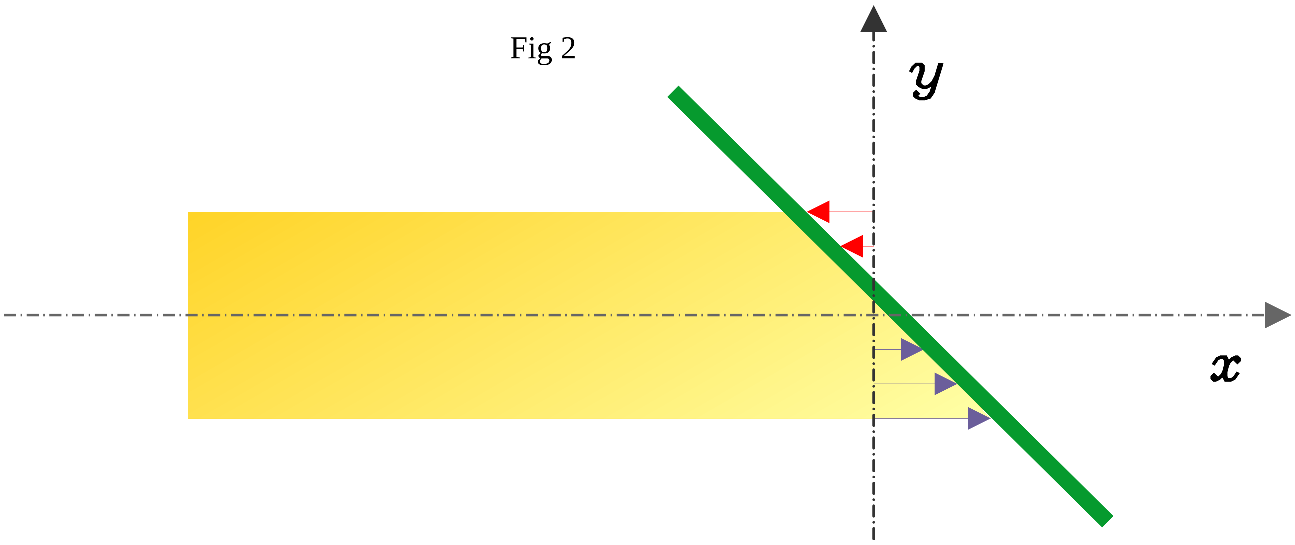


Fig 1

In our example, the upper fibres (fibres above the axis or the centre line of the beam) will be **compressed** due to compressive normal forces. The lower fibres (fibres below the axis or the centre line of the beam) will be **extended** due to tensile normal forces.

Fig 2



Bending leads to the rotation of the cross-section of the beam by an angle **Δϕ**. Bending is also characterized by **θ unitary angle of bending** where:

$$\theta = \frac{\Delta \varphi}{\Delta x}$$

The resulting normal strain is thus calculated using the formula:

$$\sigma_M = -E \theta y$$

where:

y: The distance between the axis or neutral fibre and the point M

E: Young's modulus

Normal Stress & Bending Moment

Normal stress develops at most in the farthest fibre. $\sigma_{max} = \frac{|Mf_{max}|}{\frac{I}{v}}$ where:

I: Quadratic moment (mm^4)

v: maximum value of the farthest point y_{max} (mm)

Mf_{max}: maximum value of bending moment (N.mm)

Stress Concentration

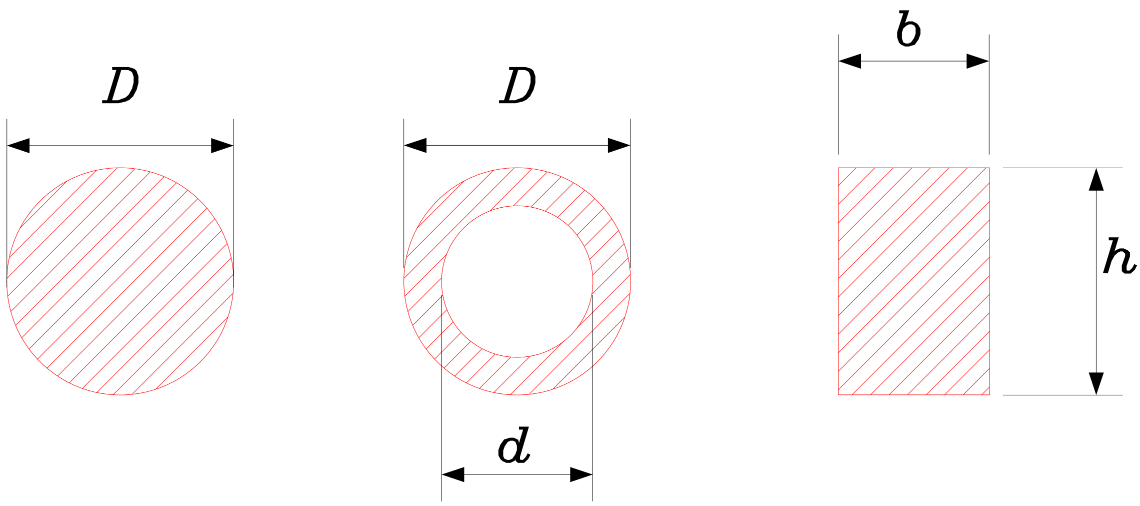
As in other solicitations, stress is more concentrated in the inconsistent cross-sections of the beam. We choose a suitable constant $\mathbf{k_f}$ from the provided abacus and we multiply it by the normal stress:

$$k_f \sigma_{max} \leq R_{pe}$$

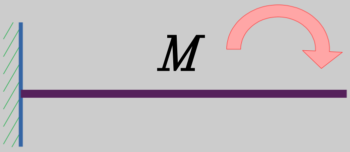
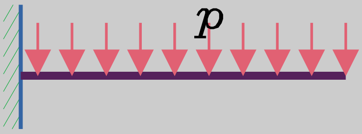
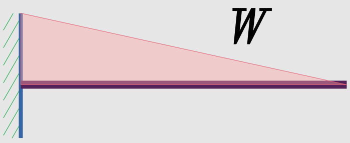
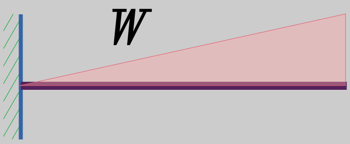
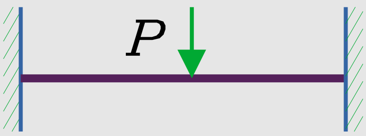
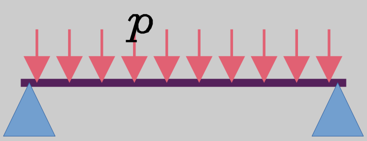
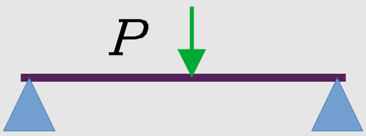
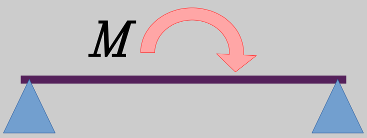
Resistance Condition

Similar to tension and compression, in order to resist failure, we have set a limit value of stress that shouldn't be surpassed. $\sigma_{max} \leq Rp_e$

This condition allows us to calculate the dimensions of our beam based on the efforts it will bear. Dimensioning depends on the geometry of the object.



$I_y = I_z = \frac{\pi D^4}{64}$	$I_y = I_z = \frac{\pi (D^4 - d^4)}{64}$	$I_y = \frac{b^3 h}{12}$ $I_z = \frac{h^3 b}{12}$

Image	Slope	Deflection
	$\theta=\frac{-ML}{EI}$	$\delta=\frac{-ML^2}{2EI}$
	$\theta=\frac{-PL^2}{2EI}$	$\delta=\frac{-PL^3}{3EI}$
	$\theta=\frac{-pL^3}{6EI}$	$\delta=\frac{-pL^4}{8EI}$
	$\theta=\frac{-WL^3}{24EI}$	$\delta=\frac{-WL^4}{30EI}$
	$\theta=\frac{-WL^3}{8EI}$	$\delta=\frac{-11WL^4}{120EI}$
	$\theta=\frac{-PL^2}{64EI}$	$\delta=\frac{-PL^3}{192EI}$
	$\theta=\frac{-pL^3}{24EI}$	$\delta=\frac{-5pL^4}{384EI}$
	$\theta=\frac{-PL^2}{16EI}$	$\delta=\frac{-PL^3}{48EI}$
	$\theta=\frac{\pm ML}{24EI}$	$\delta=\frac{\pm ML^2}{72\sqrt{3}EI}$

Composed Solicitations

In mechanical systems, different parts are exposed to various types of solicitations. For example, shafts can have axial and radial loads resulting from gears and elements of transmission, torsion due to rotation, bending due to fixtures... Loads are not uniaxial. They are not of the same nature either. We might end up with a full cohesion torsor:

$$\{\tau_{coh\,II/I}\}_G = \begin{Bmatrix} Fx & Mt \\ Fy & Mfy \\ Fz & Mfz \end{Bmatrix}_G$$

An example of composed solicitations in a shaft:

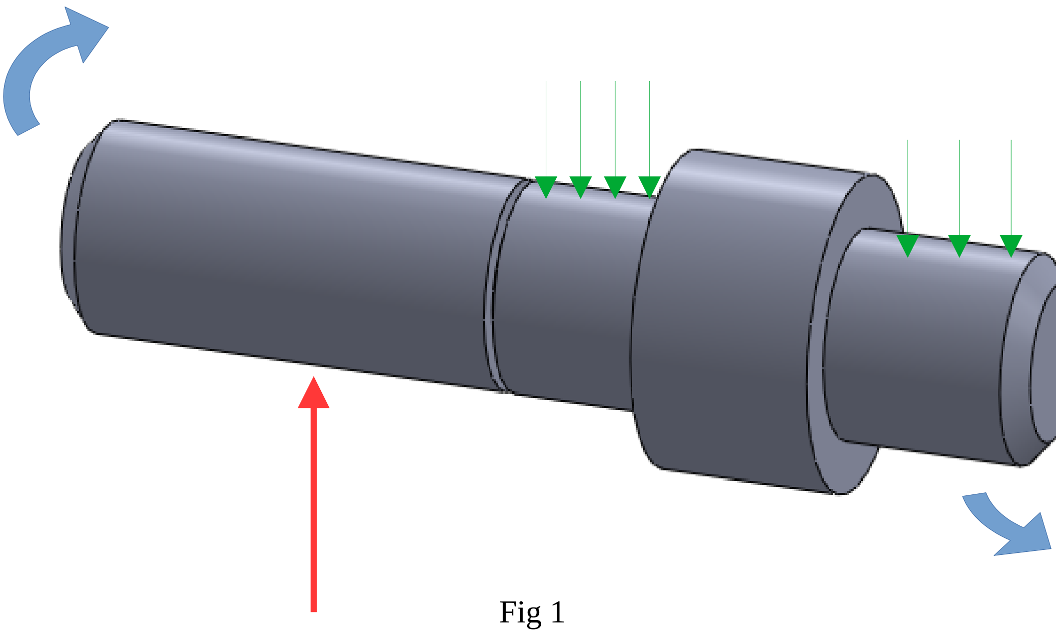


Fig 1

Examples of composed solicitations:

Case 1: Bending + Torsion:

A beam undergoes bending and torsion if the cohesion torsor in its barycentre G is written as follows:

$$\{\tau_{cohII/I}\}_G = \begin{Bmatrix} 0 & Mt \\ Fy & 0 \\ 0 & Mfz \end{Bmatrix}_G \quad \text{or} \quad \{\tau_{cohII/I}\}_G = \begin{Bmatrix} 0 & Mt \\ 0 & Mfy \\ Fz & 0 \end{Bmatrix}_G$$

Ideal Moment of Bending:

In such a solicitation, both normal and tangential stresses act simultaneously. We thus need to find the ideal moment of bending, denoted **Mf_i** in order to calculate the equivalent normal stress.

$$Mf_i = \left(1 - \frac{1}{2\lambda}\right) Mf + \frac{1}{2\lambda}\sqrt{Mf^2 + Mt^2}$$

Mf_i: ideal moment of bending (N.mm)

Mf: moment of bending (N.mm)

Mt: torque (N.mm)

λ: ratio between Rp_g and Rp_e where $\lambda = \frac{Rp_g}{Rp_e}$

Material	Steel	Cast Iron
λ	0.5	1

Resistance Condition

In order to resist failure, the beam's normal stress should not surpass a set value, where: $|\sigma_{max}| = \frac{|Mf_{max}|}{\frac{I}{v}} \leqslant Rp_e$

Deformation

There are two parameters to calculate in order to check the rigidity of the tested beam Since it is a composed solicitation, we obtain two conditions:

From bending: Maximum deflection: $\delta \leq \delta_{\lim}$.

From torsion: Angular deformation: $\theta \leq \theta_{\lim}$.

Case 2: Tension + Torsion:

A beam undergoes tension and torsion when the cohesion torsor in the barycentre of the beam is written as follows:

$$\{\tau_{cohII/I}\}_G = \begin{Bmatrix} N & Mt \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}_G$$

Ideal Stress:

Every fibre in the beam supports two types of stresses: normal and tangential. We ought to find the ideal stress to use it later in the calculations of resistance. The ideal stress is given by the formula:

$$\sigma_i = \sqrt{\sigma^2 + 4\tau^2} \quad \text{where:} \quad \tau = \frac{M_T}{J} R \quad \text{and} \quad \sigma = \frac{N}{S}$$

Resistance Condition:

For this solicitation, the resistance condition is written as follows:

$$\sigma_i \leq Rp_e \quad \text{where:} \quad Rp_e = \frac{\sigma_e}{s}$$

Case 3: Shearing + Torsion:

A beam undergoes shearing and torsion when the cohesion torsor in the barycentre of the beam is written as follows:

$$\{\tau_{cohII/I}\}_G = \begin{Bmatrix} 0 & Mt \\ Fy & 0 \\ Fz & 0 \end{Bmatrix}_G$$

Since all stresses are tangential to the cross-section, we can obtain the equivalent stress τ by summing up the tangential stresses caused by the shear forces and the twisting torque.

$$\tau_{eq} = \tau_{shearing} + \tau_{torsion} \quad \text{where:}$$

$$\tau_{torsion} = \frac{M_T}{J} R \quad \text{and} \quad \tau_{shearing} = \frac{F_{shearing}}{S}$$

$$\text{with:} \quad F_{shearing} = \sqrt{Fy^2 + Fz^2}$$

Resistance Condition:

For this solicitation, the resistance condition is written as follows:

$$\tau_{eq} \leq Rp_g \quad \text{where:} \quad Rp_g = \frac{R_g}{s}$$

Case 4: Bending + Tension/Compression:

A beam undergoes bending and tension/compression when the cohesion torsor in the barycentre of the beam is written as follows:

$$\{\tau_{cohII/I}\}_G = \begin{Bmatrix} N & 0 \\ Fy & 0 \\ 0 & Mfz \end{Bmatrix}_G \quad \text{or} \quad \{\tau_{cohII/I}\}_G = \begin{Bmatrix} N & 0 \\ 0 & Mfy \\ Fz & 0 \end{Bmatrix}_G$$

Since all stresses are normal to the cross-section, we can obtain the equivalent stress σ by summing up the normal stresses caused by the compressive or tensile forces and the bending moments.

$$\sigma_{eq} = \sigma_{tension/compression} + \sigma_{bending} \quad \text{where:}$$

$$\sigma_{bending} = \frac{Mf_{eq}}{I} R \quad \text{and} \quad \sigma_{tension/compression} = \frac{N}{S}$$

$$\text{with:} \quad Mf_{eq} = \sqrt{Mfy^2 + Mfz^2}$$

We neglect the shearing forces caused by bending, for their tangential stress is feeble when compared to normal stress.

Resistance Condition:

For this solicitation, the resistance condition is written as follows:

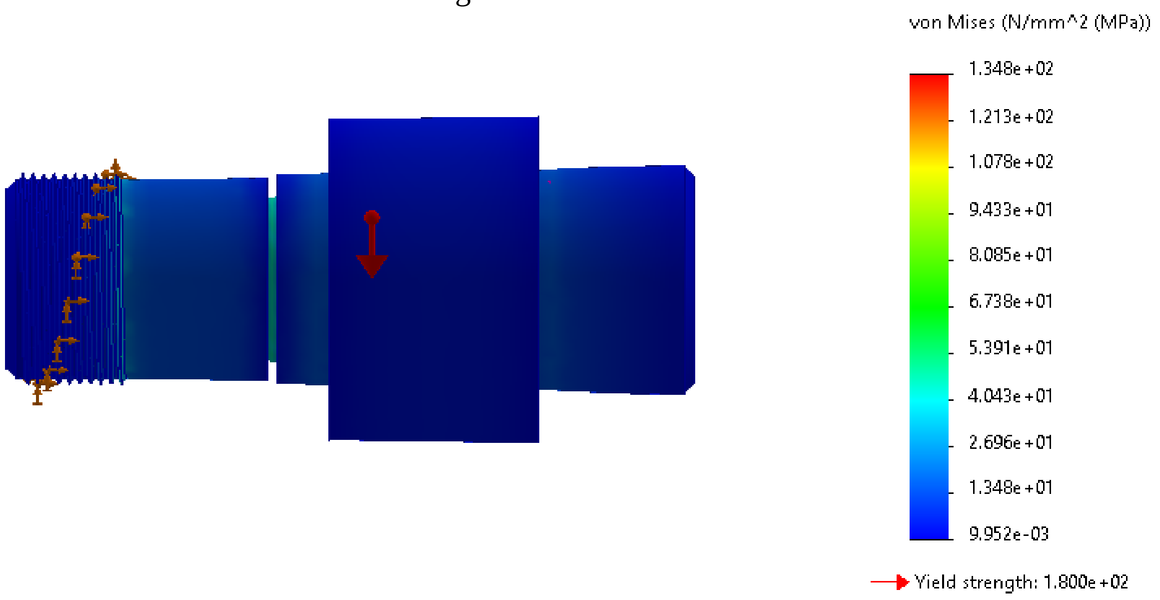
$$\sigma_{eq} \leq Rp_e \quad \text{where:} \quad Rp_e = \frac{\sigma_e}{s}$$

Generally:

A beam can undergo a variety of composed solicitations, which are not necessarily within the formerly mentioned cases. We have two criteria:

Von Mises Criterion: This criterion is the most accurate. For advanced system analysis, the software generally calculates the resulting stress using the Von Mises criterion: $\sigma_{eq} = \sqrt{\sigma^2 + 3\tau^2}$

Fig 2



Tresca's Criterion: The equivalent normal stress is calculated as follows:

$$\sigma_{eq} = \sqrt{\sigma^2 + 4\tau^2}$$

Buckling

Any long and thin beams share the same behaviour when compressive efforts tend to compress them. In fact, when the applied charge reaches a critical limit, the beam deforms, bends and fails.

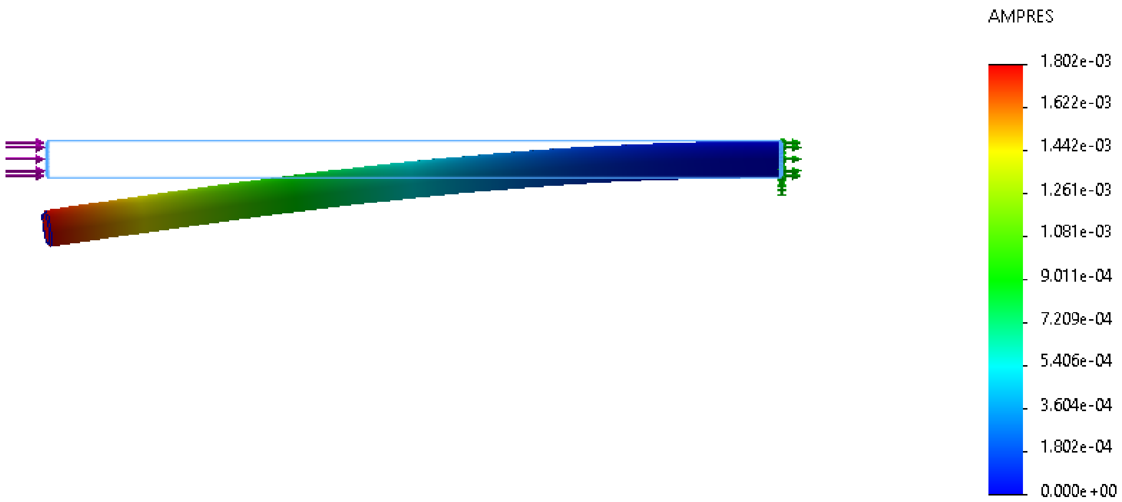


Fig 1

In this final chapter, we will deal with this phenomenon. We will learn how to wisely choose the formulae according to a set of parameters.

For an elastic material with constant mechanical properties, a beam is put to test through an application of various charges. As the charges increase, we notice a change in the behaviour of the beam's deformation. For a certain critical value F_c, the beam will fail. We distinguish 3 cases:

$F < F_c$: the beam is aligned, in a state of stable equilibrium, succumb to compression.

$F = F_c$: the beam is in a state of unstable equilibrium with a possibility of change in the latter towards a more stable equilibrium in composed bending.

$F > F_c$: the beam is unstable.

Direction of buckling:

Deformation occurs in the perpendicular direction of the weakest area moment of inertia.

Slenderness:

A beam is said to be slender if it's cross-sectional dimensions are feeble when compared to its length. Slenderness, denoted **λ** is used in the calculations of acceptable charges. We obtain it from the formula: $\lambda = \frac{L}{\rho}$

λ: the slenderness of the beam

L: unsupported length of the beam (mm)

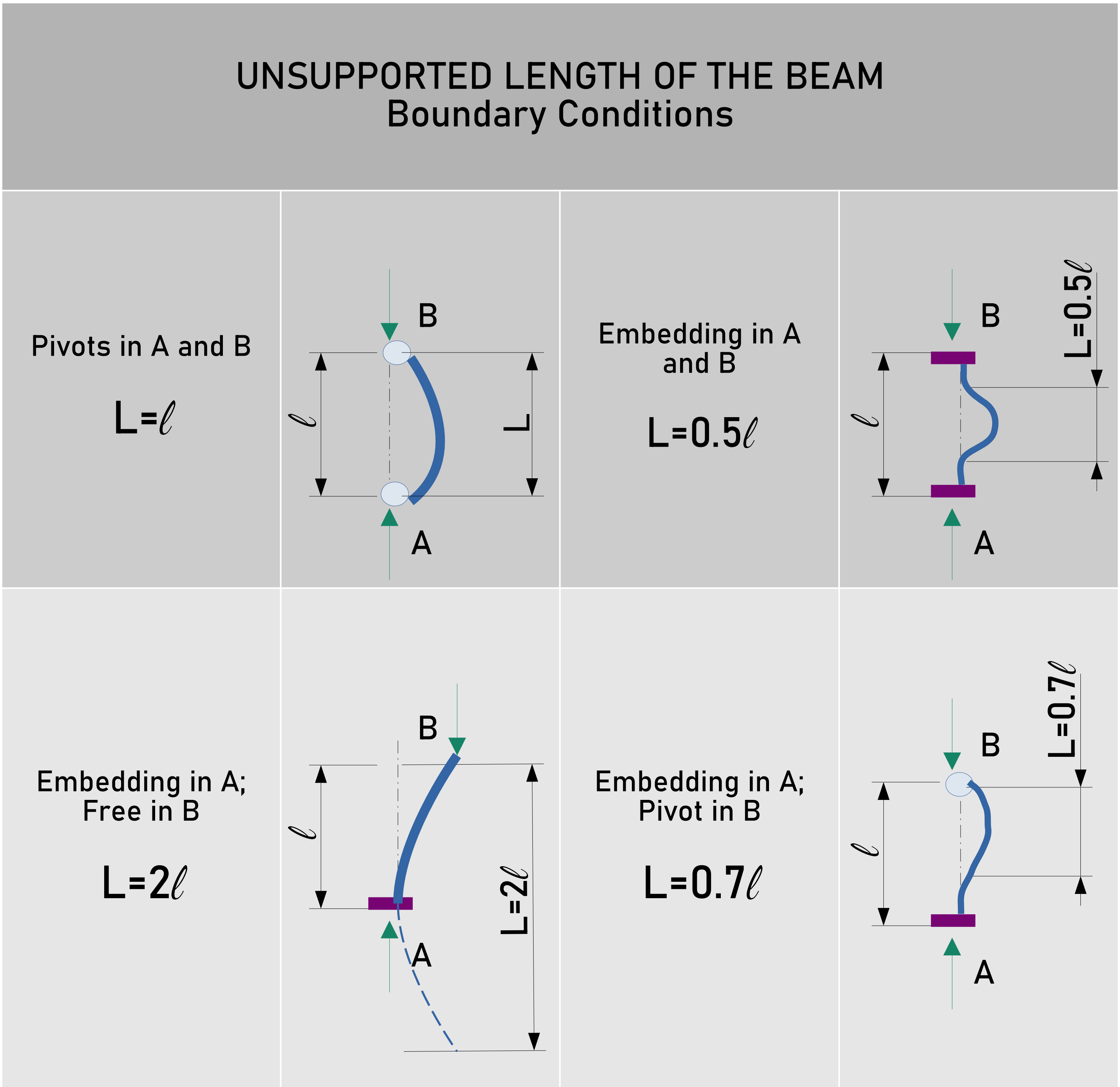
ρ: gyration radius of the cross-section (mm)

The gyration radius is obtained from the following relation: $\rho = \sqrt{\frac{I_{min}}{S}}$

I_{min}: the smallest area moment of inertia (mm^4)

S: cross-sectional area (mm^2)

Unsupported length & Boundary Conditions (Pinned, fixed & free):



Critical Force and Stress - Euler's Formula:

After calculating the slenderness ratio λ and the unsupported length L, we can get a relation between Young's modulus E and the quadratic moment of inertia I, where:

$$F_c = \frac{\pi^2 E I_{min}}{L^2} = \frac{\pi^2 E S}{\lambda^2}$$

We can also deduce the critical stress: $\sigma_c = \frac{\pi^2 E}{\lambda^2}$

Critical Slenderness:

In our calculations, we will suppose that $\sigma_c = R_e$ to work only in the elastic domain. We obtain: $R_e = \frac{\pi^2 E S}{\lambda^2}$ with $\lambda = \lambda_c \Rightarrow \lambda^2 = \frac{\pi^2 E}{R_e}$

where λ_c depends only on the material.

For steel: $\lambda_c = 100$

For aluminum: $\lambda_c = 70$

For cast iron: $\lambda_c = 60$

Security Ratio:

In other solicitations, we used a security ratio s. In buckling, this ratio is twice the regular ratio. We denote it **k** and we calculate it as follows:

$$k = 2s = \frac{2R_{ec}}{R_{pc}}$$

Resistance Condition:

Since we have learned how to calculate Euler's critical load, we know that we should never reach that value. Thus, we ought to set a new acceptable limit where danger is avoided. This new value must be strictly below the critical load. It is determined according to the value of the slenderness ratio:

Slenderness	$\lambda<20$	$20<\lambda<100$	$\lambda>100$
Beam size	Short	Average	Long
Formula	Compression	Rankine's formula	Euler's formula
	$F_{adm}=R_{pc}S$	$F_{adm}=\frac{R_{pc}S}{1+\left(\frac{\lambda}{\lambda_c}\right)^2}$	$F_{adm}=\frac{R_{pc}S}{2\left(\frac{\lambda}{\lambda_c}\right)^2}$

The Principle of Superposition

Definition: The principle of superposition is a static method used to **determine an unknown reaction** in an isostatic or 1st degree hyperstatic structures. It allows the **decomposition** of a beam, loaded with n loads, into **the sum** of n beams with only one load each.

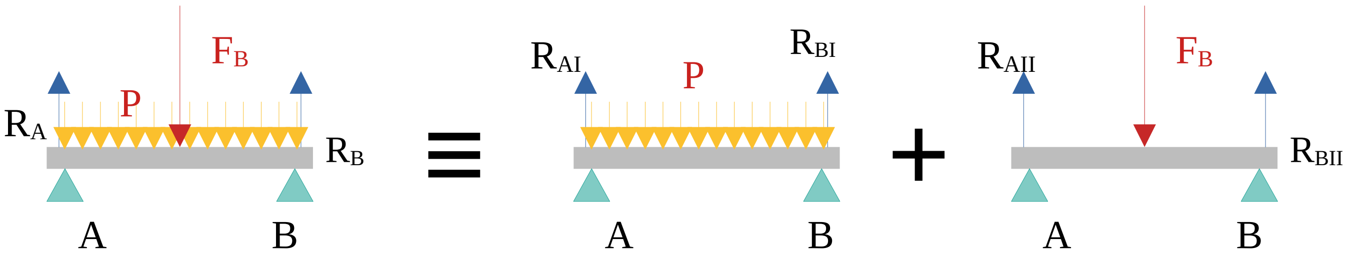


Fig 1

Degree of static indeterminacy: the number of redundant forces in the structure, and is often referred to as **the degree of hyperstaticity**. It is obtained as the result of subtracting the number of equations from the number of indeterminate reactions:

$$n_{reactions} - n_{static\ equations} = h$$

Redundant forces or reactions can't be determined by solving only the equilibrium equations. That's why the principle of superposition suggests that, in hyperstatic structures, the indeterminate force can be found using the equivalence of three equations: stress, deflection and equilibrium where in a determined point of the beam:

$$\sigma_{total} = \sigma_I + \sigma_{II}$$

$$\delta_{total} = \delta_I + \delta_{II}$$

$$R_{total} = R_I + R_{II}$$

NB: When applying the principle of superposition, boundary conditions in secondary beams are the same as the original beam.

In the given example, a simply supported beam is decomposed into simply supported beams with the same length L. If the main beam is a cantilever, the beams obtained from decomposition are also cantilevers.

Unlike the equilibrium equations and the cohesion torsors, the structure is never split in length in respect to the smaller intervals where forces and momenta are applied. The length is taken integrally.

Theorem of Castigliano

Application of Castigliano's theorem in a hyperstatistic system: This theorem generally applies to first degree hyperstatic systems. As for problems with a higher degree of indetermination, other methods are used.

Solving Method:

1. We begin with the verification of the degree of static indetermination. It should be equal to 1.
2. We decompose the initial system to obtain a set of isostatic problems: P_0 is the initial problem made isostatic by removing the hyperstatic unknown or support denoted X.
3. We superpose to P_0 the problem P_1 having the same boundary conditions of P_0 and loaded with the unknown X previously removed in the 2nd step.
4. The displacement (if X is a load) or rotation (if X is a momentum) of the application point of X is equal to zero. Thus the theorem gives: $\frac{dW_e}{dX}=0$

   The bending moment is: $M_z=M_0+M_1=M_0+X\bar{M}_1$ where:

   M_0 is the moment from the cohesion torsorsor from the problem P_0

   M_1 is the moment from the cohesion torsorsor from the problem P_1

   M_1 is the moment from the cohesion torsorsor from the problem P_1 with an assigned value equal to 1 (X = 1)

   In general: $W_e=\frac{1}{2}\int\frac{M^2}{EI}+\frac{1}{2}\int\frac{M_T^2}{GJ}+\frac{1}{2}\int\frac{N^2}{EI}+\frac{1}{2}\int\frac{V^2}{GJ}$

   After simplification:

$$X=\frac{-\int M_0\bar{M}_1dx}{\int\bar{M}_1^2}$$